QUICK COMPUTATION OF THE PARITY OF $\prod(x)$

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March 1998 (corrected on October 2001)

Let $\prod(x)$ = Number of prime numbers below or equal to x.

$|x|$=integer part of x, and $\Omega(x)$ the number of prime divisors of x (with repetition).

$\Omega(x)$ is additive, $\Omega(1) = \Omega(0) = 0$, $\Omega(p) = 1$ and $\Omega(x.y) = \Omega(x) + \Omega(y)$

$$\boxed{\Omega(x!) = \sum_{P_i \le x} \left|\frac{x}{P_i}\right| + \sum_{P_i \le x^{1/2}} \left|\frac{x}{{P_i}^2}\right| + \sum_{P_i \le x^{1/3}} \left|\frac{x}{{P_i}^3}\right| + \cdots} \qquad (1)$$

Indeed $x! = 2^{\left|\frac{x}{2}\right|+\left|\frac{x}{2^2}\right|+\cdots}.3^{\left|\frac{x}{3}\right|+\left|\frac{x}{3^2}\right|+\cdots}.5^{\left|\frac{x}{5}\right|+\left|\frac{x}{5^2}\right|+\cdots}\cdots$, therefore

$$\Omega(x!) = \left|\tfrac{x}{2}\right| + \left|\tfrac{x}{2^2}\right| + \left|\tfrac{x}{2^3}\right| + \cdots$$

$$+ \left|\tfrac{x}{3}\right| + \left|\tfrac{x}{3^2}\right| + \left|\tfrac{x}{3^3}\right| + \cdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

Summing vertically we obtain (1)

$$\boxed{\sum_{P_i \le x^{1/a}} \left|\frac{x}{{P_i}^a}\right| = \sum_{i=1}^{i=\left|\frac{x}{2^a}\right|} \prod\left[\left(\frac{x}{i}\right)^{1/a}\right]} \qquad (2)$$

Indeed $\left|\frac{x}{{P_i}^a}\right| = 1$ when $1 \le \frac{x}{{P_i}^a} < 2$ or $\frac{x}{2} < {P_i}^a \le x$ or $\left(\frac{x}{2}\right)^{1/a} < P_i \le x^{1/a}$

Therefore $\sum\limits_{P_i \le x^{1/a}} \left|\frac{x}{{P_i}^a}\right| = \sum\limits_{P_i \le \left(\frac{x}{2}\right)^{1/a}} \left|\frac{x}{{P_i}^a}\right| + \prod\left[x^{1/a}\right] - \prod\left[\left(\frac{x}{2}\right)^{1/a}\right]$

afterwards $\sum\limits_{P_i \le x^{1/a}} \left|\frac{x}{{P_i}^a}\right| = \sum\limits_{P_i \le \left(\frac{x}{3}\right)^{1/a}} \left|\frac{x}{{P_i}^a}\right| + \prod\left[x^{1/a}\right] - \prod\left[\left(\frac{x}{2}\right)^{1/a}\right] - 2\prod\left[\left(\frac{x}{3}\right)^{1/a}\right]$

and comes (2) continuing the process.

Let $\boxed{\Psi_\Omega(x) = \sum\limits_{i=1}^{i=|log_2 x|} \prod\left(x^{1/i}\right)}$ (3)

By moebius inversion we have $\boxed{\prod(x) = \sum\limits_{i=1}^{i=|log_2 x|} \mu(i)\Psi_\Omega\left(x^{1/i}\right)}$ (4)

Then we have $\boxed{\sum\limits_{i \le x} \Psi_\Omega\left(\frac{x}{i}\right) = \Omega(x!)}$ (5)

because $\sum\limits_{i \le x} \Psi_\Omega\left(\frac{x}{i}\right) = \sum\limits_{P_i \le x} \left|\frac{x}{P_i}\right| + \sum\limits_{P_i \le \left|x^{1/2}\right|} \left|\frac{x}{{P_i}^2}\right| + \sum\limits_{P_i \le \left|x^{1/3}\right|} \left|\frac{x}{{P_i}^3}\right| + \cdots$

Inverting (5) comes $\boxed{\Psi_\Omega(x) = \sum\limits_{i \le x} \mu(i)\Omega\left[\left(\frac{x}{i}\right)!\right]}$ (6)

Expanding the right part of (6) we find :

$$\Psi_\Omega(x) = \sum_{P_i \le x} \left|\frac{x}{P_i}\right| - 2* \sum_{P_i < P_j \le x} \left|\frac{x}{P_i.P_j}\right| + 3* \sum_{P_i < P_j < P_k \le x} \left|\frac{x}{P_i.P_j.P_k}\right| - 4* \sum_{P_i < P_j < P_k < P_l \le x} \left|\frac{x}{P_i.P_j.P_k.P_l}\right| + \cdots$$

Let $A_1(x) = \sum\limits_{P_i \le x} \left|\frac{x}{P_i}\right|$, $A_2(x) = \sum\limits_{P_i < P_j \le x} \left|\frac{x}{P_i.P_j}\right|$, $A_3(x) = \sum\limits_{P_i < P_j < P_k \le x} \left|\frac{x}{P_i.P_j.P_k}\right|$, $etc \cdots$

Then $\boxed{\Psi_\Omega(x) = A_1(x) - 2.A_2(x) + 3.A_3(x) - 4.A_4(x) + \cdots}$ (7)

But $\sum\limits_{i \le x} \mu(i) \left|\frac{x}{i}\right| = 1$

Also let $\boxed{x - 1 = A_1(x) - A_2(x) + A_3(x) - A_4(x) + \cdots}$ (8)

(7)-(8) result in $\boxed{\Psi_\Omega(x) = x - 1 - A_2(x) + 2.A_3(x) - 3.A_4(x) + 4.A_5(x) - \cdots}$ (9)

We note that $\Psi_\Omega(x) = x - 1$ for $1 < x < 6$ and $\Psi_\Omega(0) = 0$

So we have $\boxed{\Psi_\Omega(x) \equiv x - 1 - A_2(x) - A_4(x) - A_6(x) - \cdots [2]}$ (10)

But $$\boxed{\sum_{i\le x} Q\left(\frac{x}{i}\right) = 2(A_2(x)+A_4(x)+A_6(x)+\cdots)+2x-1} \qquad (11)$$

with $Q(x)$ number of "squarefree" $\le x$

Indeed let $\prod_k(x)$ = number of "squarefree" $\le x$ having k prime factors, we also have :

$$A_1(x) = \sum_{i\le\left\lfloor\frac{x}{2}\right\rfloor} \prod\left(\frac{x}{i}\right), \quad A_2(x) = \sum_{i\le\left\lfloor\frac{x}{6}\right\rfloor} \prod{}_2\left(\frac{x}{i}\right), \quad A_3(x) = \sum_{i\le\left\lfloor\frac{x}{30}\right\rfloor} \prod{}_3\left(\frac{x}{i}\right), etc\cdots$$

and $A_1(x)+A_2(x)+A_3(x)+A_4(x)+\cdots = \sum_{i\le\left\lfloor\frac{x}{2}\right\rfloor} \prod\left(\frac{x}{i}\right) + \sum_{i\le\left\lfloor\frac{x}{6}\right\rfloor} \prod{}_2\left(\frac{x}{i}\right) + \sum_{i\le\left\lfloor\frac{x}{30}\right\rfloor} \prod{}_3\left(\frac{x}{i}\right) + \cdots$

But $Q(x) = 1+\prod(x)+\prod_2(x)+\prod_3(x)+\prod_4(x)+\cdots$

So $$\boxed{A_1(x)+A_2(x)+A_3(x)+A_4(x)+\cdots = \sum_{i\le x} Q\left(\frac{x}{i}\right) - x} \qquad (12)$$

Let $SQ(x) = \sum_{i\le x} Q\left(\frac{x}{i}\right)$

(8) and (12) result in $2(A_2(x)+A_4(x)+A_6(x)+A_8(x)+\cdots) = SQ(x)-2x+1$, which proves (11)

and also $$\boxed{SQ(x) = \sum_{i\le x} Q\left(\frac{x}{i}\right) \equiv 1\,[2] \quad \forall x} \qquad (13)$$

(10) et (11) result in $$\boxed{\Psi_\Omega(x) \equiv \tfrac{SQ(x)-1}{2}\,[2]} \qquad (14)$$

Let $F1(x) = \sum_{i\le x}\left\lfloor\frac{x}{i}\right\rfloor$, it is easy to show that $F1(x) = 2\sum_{i=1}^{i=\lfloor\sqrt{x}\rfloor}\left\lfloor\frac{x}{i}\right\rfloor - \lfloor\sqrt{x}\rfloor^2$

Let $\boxed{F1r(x) = \sum_{i\le\lfloor\sqrt{x}\rfloor}\left\lfloor\frac{x}{i}\right\rfloor}$, so we have $F1(x) = 2.F1r(x) - \lfloor\sqrt{x}\rfloor^2$ and $\boxed{F1(x) \equiv \lfloor\sqrt{x}\rfloor\,[2]}$ (15)

But $\boxed{F1(x) = \sum_{i\le\lfloor\sqrt{x}\rfloor} SQ\left(\frac{x}{i^2}\right)}$ (16) and by moebius inversion $\boxed{SQ(x) = \sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i).F1\left(\frac{x}{i^2}\right)}$ (17)

So $SQ(x) = \sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i)\left[2.F1r\left(\frac{x}{i^2}\right) - \left\lfloor\sqrt{\frac{x}{i^2}}\right\rfloor^2\right] = 2\sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i).F1r\left(\frac{x}{i^2}\right) - \sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i).\left\lfloor\frac{\sqrt{x}}{i}\right\rfloor^2$

Let $$\boxed{SQ_1(x) = \tfrac{SQ(x)-1}{2} = \sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i).F1r\left(\frac{x}{i^2}\right) - \left(\frac{\sum_{i\le\lfloor\sqrt{x}\rfloor}\mu(i)\left\lfloor\frac{\sqrt{x}}{i}\right\rfloor^2 + 1}{2}\right)} \qquad (18)$$

Let $SR(x) = \sum\limits_{i \leq \left|\sqrt{x}\right|} \mu(i) \left|\frac{\sqrt{x}}{i}\right|^2$

$\left|\frac{\sqrt{x}}{i}\right| = 1$ for $\frac{\sqrt{x}}{2} < i \leq \sqrt{x}$, so $SR(x) = \sum\limits_{i \leq \left|\frac{\sqrt{x}}{2}\right|} \mu(i) \left|\frac{\sqrt{x}}{i}\right|^2 + \sum\limits_{i=\left|\frac{\sqrt{x}}{2}\right|+1}^{i=\left|\sqrt{x}\right|} \mu(i)$

$\left|\frac{\sqrt{x}}{i}\right| = 2$ for $\frac{\sqrt{x}}{3} < i \leq \frac{\sqrt{x}}{2}$, so $SR(x) = \sum\limits_{i \leq \left|\frac{\sqrt{x}}{3}\right|} \mu(i) \left|\frac{\sqrt{x}}{i}\right|^2 + 4 \sum\limits_{i=\left|\frac{\sqrt{x}}{3}\right|+1}^{i=\left|\frac{\sqrt{x}}{2}\right|} \mu(i) + \sum\limits_{i=\left|\frac{\sqrt{x}}{2}\right|+1}^{i=\left|\sqrt{x}\right|} \mu(i)$

Continuing the process comes $SR(x) = \sum\limits_{i=1}^{i=\left|\sqrt{x}\right|} i^2 \sum\limits_{j=\left|\frac{\sqrt{x}}{i+1}\right|+1}^{j=\left|\frac{\sqrt{x}}{i}\right|} \mu(j)$

Seeing that $x^2 \equiv 0[4]$ for even x and $x^2 \equiv 1[4]$ for odd x

we have $SR(x) \equiv \sum\limits_{\text{i odd} \leq \left|\sqrt{x}\right|} \sum\limits_{j=\left|\frac{\sqrt{x}}{i+1}\right|+1}^{j=\left|\frac{\sqrt{x}}{i}\right|} \mu(j) \quad [4]$

But $\sum\limits_{i=a+1}^{i=b} \mu(i) = M(b) - M(a)$ with $M(x)$ Moebius summatory function

So $SR(x) \equiv M\left(\sqrt{x}\right) - M\left(\frac{\sqrt{x}}{2}\right) + M\left(\frac{\sqrt{x}}{3}\right) - M\left(\frac{\sqrt{x}}{4}\right) + \cdots [4]$

But $M(x) - M\left(\frac{x}{2}\right) + M\left(\frac{x}{3}\right) - M\left(\frac{x}{4}\right) + \cdots = -1$ property of $M(x)$ for $x \geq 2$

So $SR(x) = \sum\limits_{i \leq \sqrt{x}} \mu(i) \left|\frac{\sqrt{x}}{i}\right|^2 \equiv -1[4]$ for $x \geq 4$

And (18) results in
$$\boxed{SQ_1(x) \equiv \sum_{i \leq \left|\sqrt{x}\right|} \mu(i) F1r\left(\frac{x}{i^2}\right) [2]} \qquad (19)$$

So
$$\boxed{\Psi_\Omega(x) \equiv SQ_1(x)[2]} \qquad (20)$$

And finally with (4)
$$\boxed{\boxed{\prod(x) \equiv \sum_{i=1}^{i=\left|log_2 x\right|} \mu(i) SQ_1(x^{1/i})[2]}} \qquad (21)$$

This expression allows to compute the parity of $\prod(x)$ with a complexity in $O\ (x^{1/2} * logx)$ and with a storage space in $O(1)$.